# Boolean topological graphs of semigroups

°Michał Stronkowski
•Belinda Trotta

°Warsaw University of Technology
•AGL Energy in Melbourne

BLAST, August 2013

# universal Horn classes

uH-sentences look like

$$(\forall \bar{x})\,[\varphi_1(\bar{x}) \wedge \cdots \wedge \varphi_n(\bar{x}) \to \varphi(\bar{x})],$$

or like $$(\forall \bar{x})\,[\neg\varphi_1(\bar{x}) \vee \cdots \vee \neg\varphi_n(\bar{x})]$$

where $\varphi_i(\bar{x})$, $\varphi(\bar{x})$ are atomic formulas.

uH-classes look like Mod(uH-sentences).

The uH-class generated by a class $\mathcal{K}$ equals $\mathsf{SP}^{+}\mathsf{P}_{\mathsf{U}}(\mathcal{K})$.

uH-class $\mathcal{H}$ is finitely axiomatizable (finitely based) if $\mathcal{H} = \mathrm{Mod}(\Sigma)$ for some finite set $\Sigma$ of uH-sentences.

# graph of semigroups

The graph of a semigroup $\mathbf{S} = (S, \cdot)$ is NOT a graph. It is the relational structure

$$\mathrm{G}(\mathbf{S}) = (\mathrm{S}, \mathrm{R}),$$

where

$$(a, b, c) \in R \quad \text{iff} \quad a \cdot b = c.$$

For a class $\mathcal{C}$ of semigroups let $\mathrm{G}(\mathcal{C}) = \{\mathrm{G}(\mathbf{S}) \mid \mathbf{S} \in \mathcal{C}\}$.

**Theorem (Gornostaev, S)**

Let $\mathcal{C}$ be a class of semigroups possessing a nontrivial member with a neutral element. Then $\mathrm{SP}^{+}\mathrm{P}_{\mathrm{U}}\mathrm{G}(\mathcal{C})$ is not finitely axiomatizable.

# pseudoProof

## Fact

Let $\mathcal{H}$ be a finitely axiomatizable uH-class of relational structures. Then there is a finite $n$ such that for each relational structure $\mathbf{M}$ we have

$$\mathbf{M} \in \mathcal{H} \quad \text{iff} \quad (\forall \mathbf{N} \leqslant \mathbf{M})\,[|N| \leqslant n \to \mathbf{N} \in \mathcal{H}].$$

Thus it is enough to construct for each $n$ a structure $\mathbf{M_n}$ such that

- $\mathbf{M_n} \notin \mathrm{SG}$(Semigroups),
- if $\mathbf{N} \leqslant \mathbf{M_n}$ and $|N| \leqslant n$, then $\mathbf{N} \in \mathrm{SPG}(\mathcal{C})$.

# construction of $\mathbf{M_n}$

| Elements of $M_k$ | | Elements of $Z_2^{n+6}$ | | |
|---|---|---|---|---|
| $a_0$ | | 1100 | $000 \cdots 000 \cdots 000$ | 0 |
| $a_1$ | | 0011 | $000 \cdots 000 \cdots 000$ | 0 |
| $a_0$ | $\to$ | 1010 | $000 \cdots 000 \cdots 000$ | 0 |
| $a_1$ | | 0101 | $000 \cdots 000 \cdots 000$ | 0 |
| $b$ | $\to$ | 1111 | $000 \cdots 000 \cdots 000$ | 0 |
| $c_0$ | | 0000 | $100 \cdots 000 \cdots 000$ | 0 |
| $c_1$ | | 0000 | $010 \cdots 000 \cdots 000$ | 0 |
| $\cdots$ | | | $\cdots\cdots\cdots\cdots\cdots$ | |
| $c_{k-1}$ | $\to$ | 0000 | $000 \cdots 100 \cdots 000$ | 0 |
| $c_{k+1}$ | | 0000 | $000 \cdots 001 \cdots 000$ | 0 |
| $\cdots$ | | | $\cdots\cdots\cdots\cdots\cdots$ | |
| $c_n$ | | 0000 | $000 \cdots 000 \cdots 001$ | 0 |
| $d_0$ | | 0011 | $100 \cdots 000 \cdots 000$ | 0 |
| $d_1$ | | 0011 | $110 \cdots 000 \cdots 000$ | 0 |
| $\cdots$ | | | $\cdots\cdots\cdots\cdots\cdots$ | |
| $d_{k-1}$ | | 0011 | $111 \cdots 100 \cdots 000$ | 0 |
| $d_k$ | $\to$ | 0011 | $111 \cdots 110 \cdots 000$ | 1 |
| $d_{k+1}$ | | 0011 | $111 \cdots 111 \cdots 000$ | 1 |
| $\cdots$ | | | $\cdots\cdots\cdots\cdots\cdots$ | |
| $d_n$ | | 0011 | $111 \cdots 111 \cdots 111$ | 1 |
| $d_0$ | | 0101 | $100 \cdots 000 \cdots 000$ | 0 |
| $d_1$ | | 0101 | $110 \cdots 000 \cdots 000$ | 0 |
| $\cdots$ | | | $\cdots\cdots\cdots\cdots\cdots$ | |
| $d_{k-1}$ | | 0101 | $111 \cdots 100 \cdots 000$ | 0 |
| $d_k$ | $\to$ | 0101 | $111 \cdots 110 \cdots 000$ | 0 |
| $d_{k+1}$ | | 0101 | $111 \cdots 111 \cdots 000$ | 0 |
| $\cdots$ | | | $\cdots\cdots\cdots\cdots\cdots$ | |
| $d_n$ | | 0101 | $111 \cdots 111 \cdots 111$ | 0 |
| $e$ | $\to$ | 1111 | $111 \cdots 111 \cdots 111$ | 1 |

Table. The mapping $j_k$. Elements of $Z_2^{n+6}$ are represented as words over $Z_2$. For the sake of clarity we divided these words into 3 segments of length 4, $n+1$ and 1 respectively. In the second segment $(k-1)$th, $k$th and $(k+1)$th digits are placed between dots.

# pseudoProof

### Fact

Let $\mathcal{H}$ be a finitely axiomatizable uH-class of relational structures. Then there is a finite $n$ such that for each relational structure $\mathbf{M}$ we have

$$\mathbf{M} \in \mathcal{H} \quad \text{iff} \quad (\forall \mathbf{N} \leqslant \mathbf{M})\,[|N| \leqslant n \rightarrow \mathbf{N} \in \mathcal{H}].$$

Thus it is enough to construct for each $n$ a structure $\mathbf{M_n}$ such that

- $\mathbf{M_n} \notin \mathrm{uHG}(\mathrm{Semigroups})$,
- if $\mathbf{N} \leqslant \mathbf{M_n}$ and $|N| \leqslant n$, then $\mathbf{N} \in \mathrm{uHG}(\mathcal{C})$.

### Belinda's guess

Maybe it lifts to a topological setting.

# Boolean core of a uH-class

Boolean core of $\mathcal{H}$ is

$$\mathcal{H}_{BC} = \mathsf{S}_\mathsf{c}\mathsf{P}^+(\mathcal{H}_{fin})$$

$\mathcal{H}_{fin}$ - finite structures from $\mathcal{H}$ with the discrete topology
$\mathsf{P}^+$ - the nontrivial product class operator
$\mathsf{S}_\mathsf{c}$ - the closed substructure class operator

## Example

Priestley spaces $= \mathsf{S}_\mathsf{C}\mathsf{P}^+(\{0,1\},\leqslant) = \mathsf{S}\mathsf{P}^+(\{0,1\},\leqslant)_{BC}$.

## Facts

- Every member of $\mathcal{H}_{BC}$ has Boolean topology (compact, Hausdorff, totally disconnected).
- $\mathcal{H}_{BC}$ consists of all profinite structures built, as inverse limits, from finite members of $\mathcal{H}$.

**General problem**

Axiomatize $\mathcal{H}_{BC}$ among all structures with Boolean topology.

# solution to general problem?

**Theorem (Clark, Krauss)**

Topological quasivarieties may be described by an extension of uH-logic imitating topological convergence.

But it is a nasty and awkward infinite logic.

Is there a better logic?

# standardness

$\mathcal{H}$ is standard if $\mathcal{H}_{BC}$ consists of all Boolean topological structures with reducts in $\mathcal{H}$.

If $\mathcal{H}$ is standard, then $\mathcal{H}_{BC}$ is axiomatizable by uH-theory of $\mathcal{H}$.

### Theorem (Numakura)

The variety of all semigroups is standard.

### Theorem (Clark, Davey, Haviar, Pitkethly, Talukder)

Every variety with finitely determined syntactic congruences is standard.
Examples: all varieties of semigroups, monoids, groups, rings, varieties with definable principal congruences.

### Theorem (Nešetřil, Pultr, Trotta)

Finitely generated uH-class of simple graphs is standard iff it is one of $\emptyset$, SP($\bullet$), SP($\bullet \quad \bullet$), SP($\bullet\!\!-\!\!\bullet$).

# technique for disproving standardness

A (surjective) inverse system over $\omega$ is a collection of structures $\mathbf{M}_n$, $n \in \omega$, together with (surjective) homomorphisms $\varphi_n \colon \mathbf{M}_{n+1} \to \mathbf{M}$. Its inverse limit is

$$\varprojlim \mathbf{M}_n = \{a \in \prod_{n \in \omega} M_n \mid (\forall n)\ \varphi_n(a(n+1)) = a(n)\}$$

with structure and (Boolean) topology inherited from the product

$\mathbf{M} = \varprojlim \mathbf{M}_n$ is pointwise non-separable with respect to $\mathcal{H}$ if there is a predicate $R$ and a tuple $\bar{b} \in M - R^{\mathbf{M}}$ such that for every homomorphism $\psi \colon \mathbf{M}_n \to \mathbf{N} \in \mathcal{H}$ we have $\psi(\bar{b}(n)) \in R^{\mathbf{N}}$.

**Theorem (Clark, Davey, Jackson, Pitkethly)**

Assume that $\mathbf{M} = \varprojlim \mathbf{M}_n$, a surjective inverse limit of finite structures, is pointwise non-separable with respect to $\mathcal{H}$ and every $n$-element substructure of $\mathbf{M}_n$ is in $\mathcal{H}$. Then $\mathcal{H}$ is non-standard.

**Theorem (S, T)**

Let $\mathcal{H} = \mathsf{SP}^+\mathsf{P}_\mathsf{U}\mathsf{G}(\mathcal{C})$ be the uH-class generated by a class $\mathsf{G}(\mathcal{C})$ of graphs of semigroups possessing a nontrivial member with a neutral element. Then $\mathcal{H}$ is non-standard - $\mathcal{H}_{BC}$ is not definable in uH-logic.

**pseudoProof**

Structures $\mathbf{M}_n$ from non-finite axiomatization proof may be slightly modified and connected by homomorphism, thus giving a needed inverse system.

# first order definability

Maybe $\mathcal{H}_{BC}$ is fo-definable?

**Example (Clark, Davey, Jackson, Pitkethly)**

Let $\mathbf{L}$ be a finite structure with a lattice reduct. Then $\mathsf{S_cP}(\mathbf{L})$ is first order definable. But there are some non-standard $\mathsf{S_cP}(\mathbf{L})$.

**Example (Stralka, Clark, Davey, Jackson, Pitkethly)**

Priestley spaces form a non-fo definable class.

**pseudoProof**

Because there exists Stralka space $(C, \leqslant)$:
$C$ - Cantor space
$\leqslant$ - cover or equal relation

$(C, \leqslant)$ is a union of copies of $(\{0\}, =)$ and $(\{0, 1\}, \leqslant)$
but it is NOT a Priestley space.

# techniques for disproving fo-definability

A topological space is a $\lambda$-space, $\lambda \in \mathbb{N}$, if it is a disjoint union of at most $\lambda$ pieces each of which is either a one point or one point compactification of a discrete topological space.

**Theorem (Clark, Davey, Jackson, Pitkethly)**

Let $\mathcal{H}$ be non-standard, witnessed by **M** (**M** has Boolean topology an the relational reduct in $\mathcal{H}$). If

- up to isomorphism, **M** has only finitely many connected components and all them are finite ($1^{st}$ technique)

or

- **M** has a $\lambda$-topology + some technical condition ($2^{nd}$ technique)

then $\mathcal{H}_{BC}$ is not fo-definable.

# lack of fo-definablility

### Theorem (S, T)

Let $\mathcal{H} = \mathsf{SP}^{+}\mathsf{P}_{\mathsf{U}}\mathsf{G}(\mathcal{C})$ be the uH-class generated by a class $\mathsf{G}(\mathcal{C})$ of graphs of semigroups possessing a nontrivial member with a neutral element. Then $\mathcal{H}_{BC}$ is not fo-definable.

### pseudoProof

- If $(\{0,1\},\vee) \in \mathcal{C}$, then $1^{st}$ technique applies to a modification of Stralka space.
- If $(\mathbb{Z}_k,+) \in \mathcal{C}$ or $(\mathbb{N},+) \in \mathcal{C}$, then $2^{nd}$ technique applies to **M** constructed for disproving standardness.

### General problem

Axiomatize $\mathcal{H}_{BC}$ among all structures with Boolean topology.

What about monadic second order logic?

This is all Thank you!